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Working report: Research upon Directed Percolation model by introducing tensor network technique

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ABSTRACT

In this short-term research activity, we are dedicated to attack the problem of directed percolation(DP) model with the help of nowadays more advanced tensor network technique. For starting, we constructed the proper tensor network representation which is equivalent to the scenario described by DP model. Then we will strive to deal with our tensor network with proper tensor projection technique, which we are still on the way of testing and making programs. The essential nature of asymmetric property of DP model seems to hinder the pursuit of correct renormalization group(RG) flow, but it could be also the intriguing part of this problem. Our ultimate goal will be to catch the right RG flow, which we believe is highly-related to the critical exponents of the DP model.

Percolation means to make liquid to pass through fine interstices and is often used in the context of filtering. This is an important issue to deal with since this model is related to our daily lives such as to predict the clogging of pipes, etc. There are two different ways of percolations. The first one is the undirected percolation, meaning that the liquid once entering the substance, it will spread isotropically and the agent can pass through any channels in any directions. While the liquid is spreading in the pattern of directed percolation(DP), there will be one chosen direction for the agent and therefore it can only spread along the assigned pathway. The isotropic percolation problem has been solve exactly with a correct set of critical exponents. However, DP case is not analytically solvable and therefore needs the help from the numerical side. From now on, I will focus only on the DP problem.

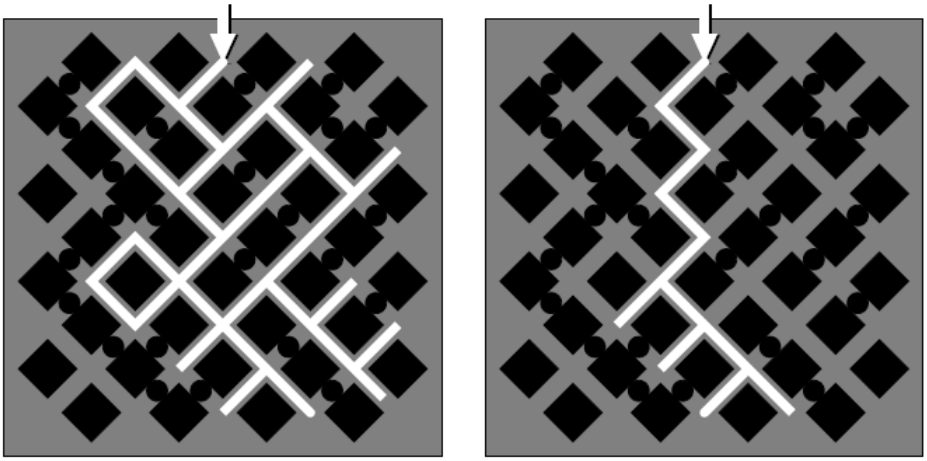

Figure 1. Two plots showing the difference of isotropic percolation(left) and directed percolation(right). On the left one can see once the agent enter the system, it begins to spread in isotropic way. But the right panel illustrates that in DP process, the flow of agent is restricted to an uni-direction.(figure source: Non-equilibrium Phase Transitions Volume I: Absorbing Phase Transition by M. Henkel, H. Hinrichsen, and S. Lübeck(Springer))

To visualize the model, we can imagine that there are numbers of lattice sites spanned in a N dimensional space. Each lattice is connected to neighbor lattices with a channel. For each channel, there is a percolation probability p which determines

the chance for the agent going through this channel to neighboring site. p serves as the parameter for phase transition. When the probability is larger than the critical value p_c, then the system can maintain active with non zero particle density. However, if the probability is lower than p_c, the system will be clogged and ends up no particle after certain numbers of steps. This final phase, which the system can only enter but cannot escape, is called the absorbing phase. It also means that there is not any detailed balance between active and absorbing states, showing that it is in fact a non-equilibrium phase transition.

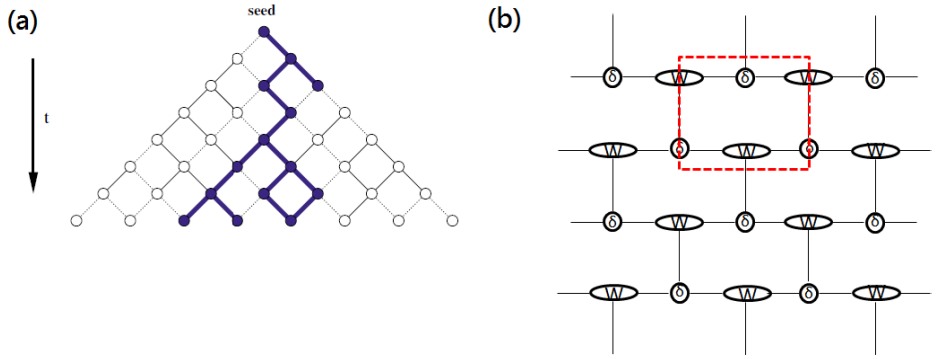

Figure 2. (a) shows the simplified scenario of DP model with lattices and bonds that connect between sites. (b) the tensor network representation of DP model. Region enclosed by red dotted line is the minimum repeating block to form the whole bulk tensor. Therefore, we will perform singular value decomposition for the right and left parts. (figure source(a): Non-equilibrium Phase Transitions Volume I: Absorbing Phase Transition by M. Henkel, H. Hinrichsen, and S. Lübeck(Springer))

The reason why this is such an important issue is, as described earlier, that this model cannot be solved exactly. To approach this issue, we need to rely on other technique such as Monde Carlo method. But despite the convenience of using Monde Carlo which could give out the correct numbers of particle density with only few lines of programming needed, it could not reveal the critical exponents, which are very important in understanding a thermodynamic system for physicists. Therefore, we turn to another nowadays more mature numerical approach, the tensor network technique.

To start with, we need to interpret the DP process into tensor network representation. As shown in Fig. 2(a), the DP process is plotted as a combination of lattice sites and bonds between them. Slim solid(dotted) line means the path is open(closed). For a single bond the possibility to be open is ρ and on the contrary $1-\rho$ for being closed. The hard purple lines represent the ultimate pathway of the agent transfer. For such system, we could translate it into tensor network language as presented in Fig. 2(b). For each indices, the dimension is equal to 2 and δ tensor is a tensor of Kronecker delta function:

$$\delta_{ijk} = \begin{cases} 1, & i=j=k. \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

while the W tensor corresponds to the tunneling of agent between nearby sites. The tensor elements of W tensor are summerized in Fig.3.

After the construction of tensor network representation, we are able to obtain the results of any DP precesses for any lattice size with any time steps...in an ideal world where calculation resource is infinite. Once we perform the contraction of tensor, in each time it will enlarge the bond dimension from χ to χ^2 and it ends up a tensor with enormous number of tensor elements which is beyond the capacity of computation memory. Therefore, we need to perform tensor projection technique in each step of contraction to reduce the bond dimension. There are several ways of having this done. One of these is, as shown in Fig.4(a), the so-called tensor renormalization group(TRG) technique or as in ref.[1] TRG with the help of higher order singular value decomposition(HOSVD), the higher order tensor renormalization group(HOTRG). The idea of HOTRG is not difficult. Supposed we have a tensor after the n^{th} RG step, $T^{(n)}$, then the contracted tensor in y direction is:

$$M^{(n)}_{xx'yy'} = \sum_i T^{(n)}_{x_1 x'_1 y i} T^{(n)}_{x_2 x'_2 i y'} \tag{S1}$$

now we can see that after contraction, the bond dimension of $M^{(n)}$ is equal to χ^2 in x direction. Therefore, we have to insert the projection for reducing the bond dimension to χ again. The way of doing this is to perform the HOSVD upon $M^{(n)}$:

$$M^{(n)}_{xx'yy'} = \sum_{ijkl} S_{ijkl} U^L_{xi} U^R_{x'j} U^U_{yk} U^D_{y'l} \tag{S1}$$

Real configuration	i	j	k	Tensor element
	0	0	0	1
	1	0	0	$1-p$
	0	1	0	$1-p$
	0	1	1	p
	1	0	1	p
	1	1	0	$(1-p)^2$
	1	1	1	$p(2-p)$

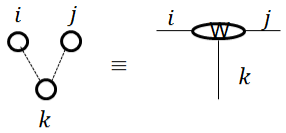

Figure 3. The tensor elements of W tensor and their correspondence to modes of DP model. Tensor elements not shown in this chart are equal to zero.

Because we will have to insert the projection operator in x direction, we keep only U^R and U^L and compare the sum of their singular values within χ. We then preserve the one with larger summation of singular values and use it as the projector in both sides. After contraction with the projectors, we can obtain $T^{(n+1)}$(Fig.4(a)). This method is very effective in dealing with physical models of higher symmetry such as 2D Ising model. However, our DP process has a specific direction of flow and therefore results in an asymmetry of the block tensor. After a heavy trial of HOTRG into our system, we figured out that the asymmetric nature will always make HOTRG process unreliable. As a result, we seek to another method, which is inspired by the infinite projected-entangled pair state(iPEPS) ansatz.

As shown in Fig.4(b), the most important issue of iPEPS calculation is to have the correct renormalization group of corner-transfer-matrix which is related to the tensor environment. In ref.[2], they present a good way of having this done by the help of QR decomposition. The key idea is that now we could obtain our desired projectors with tensor R and its inverse tensor, R^{-1}. As in Fig.4(b), we contract R and R^{-1} and then perform singular value decomposition(SVD). We then preserve until the χ^{th} largest singular values and discard the rest. Next, we can have the projectors after contracting R, $V(U)$, and $s^{-1/2}$. The merit of this non-unitary projection is that now we do not have to choose between $U^R(U^U)$ and $U^L(U^D)$ because the projection is cast upon a contracted bond connecting nearby tensors. So the effect of asymmetry of tensor block of our bulk tensor will be minimized.

In sum, the current status of this project is now at a stage of confusion since we have to pay extra attention in dealing with DP model, especially there is no analytical answer that we can compare with. But we also believe this can be an important problem not only because of its non-equilibrium nature, but also due to its connection to real life issues. We plan to continue the research even though I have been returned to Taiwan after two months of period. In the future, we expect to first construct a compact tensor regime that can interpret DP model in a proper way, with the help of other advanced tensor network technique [3-5]. Our ultimate goal is to catch the correct RG flow and then calculate the correct set of critical exponents, which is not accessible in an analytical approach.

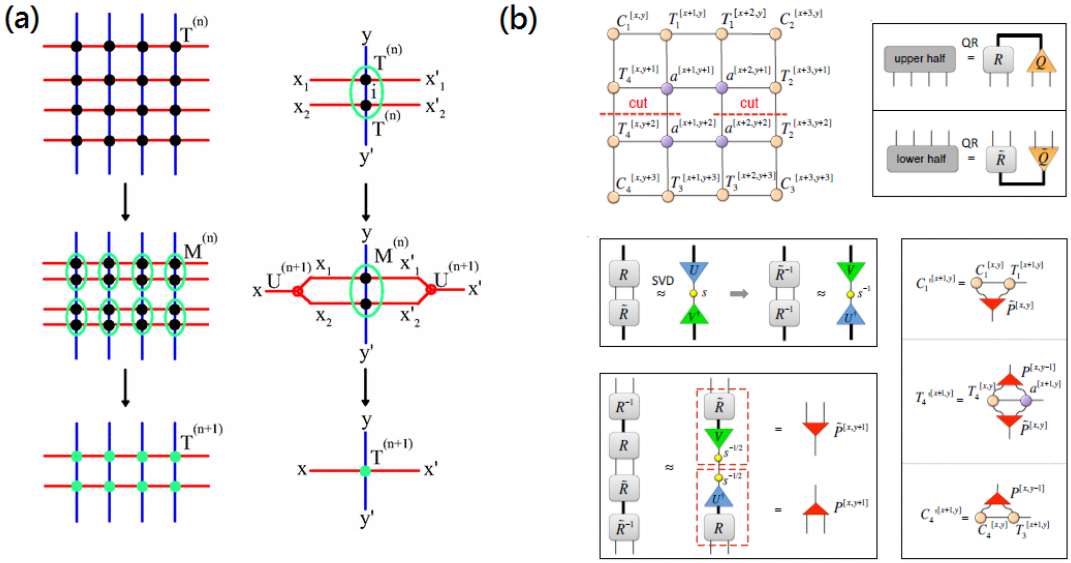

Figure 4. Two candidates for performing tensor projection of our DP bulk tensor: (a) the HOTRG process [1]. (b) the non-unitary projection inspired by the same process of corner-transfer-matrix method from iPEPS [2].

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